Lecture 10. Equations of Motion

Centripetal Acceleration, Gravitation and Gravity

The centripetal acceleration of a body located on the Earth’s surface at a distance \( \vec{r} \) from the center is the force (per unit mass) that makes the body circulate around the axis of the Earth with an angular velocity \( \vec{\Omega} \). Remember that we are learning about the forces per unit mass (accelerations) produced on the ocean by the Earth’s rotation. Essentially, these are forces that arise in the transformation from a fixed reference frame on Earth to an inertial reference frame (as viewed from the stars). These forces per unit mass, which are considered apparent forces, are the Coriolis acceleration and the centripetal acceleration. Therefore, the acceleration of a fluid parcel in the ocean with respect to an inertial frame can be described as:

\[
\frac{d\vec{u}}{dt} = -2 \vec{\Omega} \times \vec{u} - \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) + g_f - \frac{1}{\rho} \nabla p + \text{other Forces},
\]  

(10.1)

where the subscripts (\( e \)) indicating properties in a non-inertial reference frame have been dropped, and where the gravitational acceleration \( g_f \) has been added as one of the other forces per unit mass. The gravitational acceleration arises from the attractive force between masses \( M_1 \) and \( M_2 \) separated by a distance \( r \), or Newton’s law of Gravitation. This is a force that acts along an imaginary line that connects \( M_1 \) (Earth’s center) to \( M_2 \) (an object on the Earth’s surface). In (10.1), the terms \( g_f - \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \) are referred to as the acceleration due to gravity, or gravitational acceleration modified by the centrifugal (or negative centripetal) acceleration, as illustrated in the following figure.

Thus, the acceleration due to gravity \( \vec{g} \) represents a vector whose only non-zero component is in the vertical direction or \( (0, 0, -g) \). Notice also that \( g \) varies with latitude because the
centrifugal acceleration, whose magnitude is $\Omega^2 r \cos \lambda$, is zero at the poles and maximum at the equator. This distribution of the centrifugal acceleration with latitude, together with the fact that $r$ is longer at the equator than at the poles, makes $g$ be maximum at the poles and minimum at the equator. The variation is approximately 0.5% so that for all practical purposes, we will take $g$ as a constant that equals 9.80 m/s$^2$. Hence, the equations of motion now look like

$$\frac{d\mathbf{u}}{dt} = -2\Omega \times \mathbf{u} + \mathbf{g} - \frac{1}{\rho} \nabla p + \text{other Forces.} \quad (10.2)$$

Among the other forces, we have the frictional forces to be studied next.

**Friction**

Friction tends to retard the flow, i.e., it tends to decelerate the flow due to the interaction of certain portions of a flow with other portions or with a solid boundary. This interaction causes the flow to develop shears or gradients. When the wind acts on the sea surface, the friction presented by the latter retards the wind velocity at the sea surface but accelerates the water. These types of frictional forces are illustrated in the following figure.

As you can see, friction produces flow shears and hence, a flux of momentum (per unit area) from the regions of fast flow to the regions of slow flow, or shear stresses. We will study friction (or shear stresses) at molecular scales and at turbulent scales. First, let us look at the representation of shear stresses at molecular scales.

Stress has units of force per unit area or pressure (Pascals, 100 Pa = 1 mb). Shear stresses $\tau$ are
proportional to the rate of shear normal to the surface on which stress is exerted \((\partial u/\partial z)\), i.e.,

\[
\tau_x = -\mu \frac{\partial u}{\partial z}
\]

The proportionality coefficient \(\mu\) is the molecular dynamic viscosity, which has typical values for water of \(10^3\) kg m\(^{-1}\) s\(^{-1}\), and is a property of the fluid, e.g., it is different for water and for oil. Each component of the flow can experience shear stresses that are composed of three parts that coincide with each direction of our right-handed coordinate system. One part, as described above for the \(u\) component of the flow, is proportional to \(\partial u/\partial z\). The other two parts are proportional to \(\partial u/\partial x\) and \(\partial u/\partial y\), respectively, as shown in the following figure in which the volume element has been used again to illustrate these concepts.

Then the force per unit mass exerted by molecular shear stresses on the \(u\) component of the flow is

\[
F_x = \frac{1}{\rho} \left[ \frac{\partial}{\partial x} (\mu \frac{\partial u}{\partial x}) + \frac{\partial}{\partial y} (\mu \frac{\partial u}{\partial y}) + \frac{\partial}{\partial z} (\mu \frac{\partial u}{\partial z}) \right] = \frac{\partial}{\partial x} (\nu \frac{\partial u}{\partial x}) + \frac{\partial}{\partial y} (\nu \frac{\partial u}{\partial y}) + \frac{\partial}{\partial z} (\nu \frac{\partial u}{\partial z})
\]

where \(\nu = \mu/\rho\) is the coefficient of kinematic molecular viscosity with typical values for water of \(10^6\) m\(^2\)/s. Note that if \(\nu\) is considered constant, which is a rough assumption to simplify the problem,

\[
F_x = \nu \nabla^2 u
\]

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where $\nabla^2$ is the Laplacian operator $\left( \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2} \right)$. Hence, with the concepts studied thus far, the equations of motion, can be written as:

$$\frac{du}{dt} + 2 \overrightarrow{v} \times \overrightarrow{u} = \overrightarrow{g} - \frac{1}{\rho} \nabla p + (v \nabla^2 \overrightarrow{u}). \quad (10.3)$$

These are the so-called **Navier-Stokes** equations. These equations exclude any possibility of local fluctuations in flow velocity and presuppose that the flow is laminar. This assumption is probably quite restrictive for flows in the ocean as will be shown next.

If we compare the non-linear (also known as advective or inertial) terms in (10.3) to molecular friction, we get an idea of the importance of molecular friction (or viscosity) on the flow. The non-linear terms, e.g., $\overrightarrow{u} \frac{\partial u}{\partial x}$ can be scaled as $U^2/L$ where $U$ is a typical velocity and $L$ is a typical length scale. In turn, molecular friction can be scaled as $vU/L^2$. The ratio of inertial to viscous effects is

$$\frac{U^2/L}{vU/L^2} = \frac{UL}{v} = Re,$$

which is the non-dimensional Reynolds number. The flow is said to be **laminar** when $Re < 1000$. It is in transition to turbulence when $100 < Re < 10^6$ to $10^8$. And it is turbulent when $Re > 10^8$ unless the fluid is stratified. For instance, if we consider an ocean flow with typical speed of 0.1 m/s, typical length scales of 1 km, and $v$ equals $10^{-6}$ m$^2$/s, then $Re = 10^4$. In this case, the non-linear (or inertial) effects are very strong compared to molecular viscosity effects. Now, does this mean that frictional effects are negligible in the ocean? The answer is NO, frictional stresses due to turbulence are significant enough to alter oceanic currents. Molecular friction is usually negligible at scales greater than one meter or so. However, it is relevant at smaller scales as in biogeochemical processes in the sediments. In this course, we are studying oceanic physical processes with minimal scales of a few hundreds of meters, which are usually turbulent (very high $Re$). Therefore we will now focus our attention on turbulent flows.

**Friction due to turbulence in the equations of motion**

In rough terms, turbulent properties (e.g. pressure, flow, temperature, salinity) have a mean ($\overline{u}$) and a fluctuating part around the mean ($u'$). (Note that in this section, the overbar denotes an averaged quantity, not a vector as in previous sections.) To illustrate this concept, consider the following figure of water temperature variations in time, where the brackets ($\langle \rangle$) denote average over time:
The total temperature signal consists of a mean plus a fluctuating part. Note first that the time average of the mean part equals the mean. Then note that the time average of the fluctuating part is zero because the negative fluctuations integrated over time must equal the positive fluctuations. Because of this, the time average of the product of the mean part times the fluctuations is zero. However, the time average of the product of the fluctuations times the fluctuations is positive and non-zero.

With these ideas in mind, let us now consider turbulent flows in the equations of motion (10.2). We must consider the mean and fluctuating part of each dependent variable (u, v, w, p, ρ) and average over time. This procedure yields the same terms in (10.2), which now describe mean quantities, and additional terms that involve products of the fluctuating parts, which arise from the non-linear or advective terms. This will be explained next.

Using the flow mean and fluctuating parts, and substituting them in the advection terms of the u-momentum equation after averaging over time yields:

\[ \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} + \bar{w} \frac{\partial \bar{u}}{\partial z} + \langle u' \frac{\partial u'}{\partial x} \rangle + \langle v' \frac{\partial u'}{\partial y} \rangle + \langle w' \frac{\partial u'}{\partial z} \rangle \]

The first three terms belong in \( \frac{\partial \bar{u}}{\partial t} \). The other three terms can be written as:
\[
\frac{\partial}{\partial x} \langle u' u' \rangle + \frac{\partial}{\partial y} \langle u' v' \rangle + \frac{\partial}{\partial z} \langle u' w' \rangle, \tag{10.4}
\]

because

\[
\frac{\partial}{\partial x} \langle u' u' \rangle + \frac{\partial}{\partial y} \langle u' v' \rangle + \frac{\partial}{\partial z} \langle u' w' \rangle = \langle u \frac{\partial u'}{\partial x} \rangle + \langle v \frac{\partial u'}{\partial y} \rangle + \langle w \frac{\partial u'}{\partial z} \rangle + \langle u \frac{\partial v'}{\partial x} \rangle + \langle v \frac{\partial v'}{\partial y} \rangle + \langle u \frac{\partial w'}{\partial z} \rangle
\]

and the fourth, fifth and sixth terms on the right hand side, altogether = 0, i.e.,

\[
\left[ \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} \right] = 0
\]

due to continuity. The terms \( \langle u' u' \rangle, \langle u' v' \rangle, \langle u' w' \rangle \) in (10.4) are called the Reynolds stresses per unit mass and volume and represent additional variables to be included in the equations of motion. **The Reynolds stresses arise from the advective (or non-linear or inertial) terms** and are found (mostly through laboratory experiments) to be proportional to the gradients of the mean flow, i.e.,

\[
\langle u' u' \rangle = -A_x \frac{\partial \bar{u}}{\partial x}, \quad \langle u' v' \rangle = -A_y \frac{\partial \bar{u}}{\partial y}, \quad \langle u' w' \rangle = -A_z \frac{\partial \bar{u}}{\partial z}
\]

This relation of the fluctuating part of the turbulent flow to the mean turbulent flow is called a **turbulence closure**. The proportionality constants \((A_x, A_y, A_z)\) are the eddy (or turbulent) viscosities and vary according to the flow characteristics as will be seen later. The closure allows the solution of the equations of motion by reducing the number of variables and having the same number of variables and equations. Then, the friction terms in (10.2) can be expressed as the Product/unit area generated by shear stresses on the flow. For the \(u\)-momentum equation, the "friction term," which represents mixing, can be written as:

\[
F_x = \frac{\partial}{\partial x} \left[ A_x \frac{\partial \bar{u}}{\partial x} \right] + \frac{\partial}{\partial y} \left[ A_y \frac{\partial \bar{u}}{\partial y} \right] + \frac{\partial}{\partial z} \left[ A_z \frac{\partial \bar{u}}{\partial z} \right] \quad \tag{10.5}
\]

The eddy viscosities have units of \([L^2/t]\) and vary in space and time. Their values depend on the flow shears and on stratification conditions, i.e., on the Richardson number (to be studied later). Typical values for the horizontal eddy viscosities, \(A_x\) and \(A_y\), range between \(10^3\) and \(10^5\) m²/s.
Typical values for the vertical eddy viscosity, $A_z$, oscillate between $10^{-5}$ and $10^1$. Although $A_z$ is orders of magnitude smaller than $A_x, A_y$, the forces involving $A_z$ are frequently stronger because of the greater shears that develop in the vertical direction. Note that these turbulent or eddy viscosities are up to $10^3$ times greater than the molecular viscosities and therefore are dominant in the momentum balance. For the $y$ and $z$ components of the friction term, $v$ and $w$ replace $u$ in (10.5).

Then, the equations of motion or momentum equations in full form, where the bars denoting averaged quantities have been dropped, are:

$$\frac{du}{dt} - f v = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\partial}{\partial x} [A_x \frac{\partial u}{\partial x}] + \frac{\partial}{\partial y} [A_y \frac{\partial u}{\partial y}] + \frac{\partial}{\partial z} [A_z \frac{\partial u}{\partial z}]$$

$$\frac{dv}{dt} + f u = - \frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\partial}{\partial x} [A_x \frac{\partial v}{\partial x}] + \frac{\partial}{\partial y} [A_y \frac{\partial v}{\partial y}] + \frac{\partial}{\partial z} [A_z \frac{\partial v}{\partial z}]$$

$$\frac{dw}{dt} - 2\Omega \cos \lambda u = - \frac{1}{\rho} \frac{\partial p}{\partial z} - g + \frac{\partial}{\partial x} [A_x \frac{\partial w}{\partial x}] + \frac{\partial}{\partial y} [A_y \frac{\partial w}{\partial y}] + \frac{\partial}{\partial z} [A_z \frac{\partial w}{\partial z}]$$

Note that the $w$ component of the momentum equations reduces to the hydrostatic approximation as most of the terms in that component are several orders of magnitude smaller than $g$. Then, the $w$ momentum equation is:

$$\frac{\partial p}{\partial z} = -\rho g$$

Remember that these $u, v, w$ momentum equations represent mean turbulent motion. Together with the equations of state, mass conservation, salt conservation, and heat conservation, we form the set of 7 equations and 7 unknowns ($u, v, w, p, S, T, \rho$) that describes the dynamics of any part of the ocean.